Chapter 4

Wednesday, June 16, 2021 2:01 PM

2) $B_1, B_2, \ldots, B_n$

$0 < P(B_i) < 1$

$\bigcap_{j=1}^{n} A_j \neq \emptyset$ $A_j = B_j$ or $A_j = B_j^c$

$\geq 2^n$ elements

$P\left(\bigcap_{j=1}^{n} A_j\right) = \prod \underbrace{P(A_j)}_{>0} > 0$

3) $P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} P\left(\bigcap_{n=1}^{N} A_n\right) = \lim_{N\to\infty} \prod_{n=1}^{N} P(A_n) = \prod_{n=1}^{\infty} P(A_n)$

continuity of probability.

5) $g(X) \neq C \Rightarrow$

$\exists\ a : 0 < P(\underbrace{g(X) > a}_{E}) < 1$

g is measurable

$P(E) = P(E \cap E) = P(E)^2$

10) ? (X_n) - independent

$P(\sup X_n < \infty) = 1 \Leftrightarrow \exists M \sum P(X_n > M) < \infty$?

$(\Leftarrow)$ Borel - Cantelli: $P(\#\{n : X_n > M\}$ - finite$) = 1$.

w/o independence.

$P(\sup X_n < \infty) = 1$

$(\Rightarrow)$ Assume $\forall M$: $\sum P(X_n > M) = \infty$

Independent Borel - Cantelli:

Independent Borel-Cantelli:

$\forall M$. $P(\underbrace{\{\#\{n : X_n > M\} - \text{infinite}\}}_{A_M}) = 1$

$P(\bigcap_{M=1}^{\infty} A_M) = 1$

$\quad\quad\quad\quad \|$

$\{\forall M : \#\{n : X_n > M\} - \text{infinite}\}$

$\quad\quad\quad\quad \Downarrow \quad (\Uparrow)$

$\sup X_n > M \;\; \forall M \implies \sup X_n = \infty$

$\{\sup X_n < \infty\} = \bigcup_{M=1}^{\infty} \{\sup X_n < M\}$

$\lim_{M \to \infty} P(\{\sup X_n < M\}) = 1$

Bonus: Construct (X_n) independent,

(+2) $P(\sup X_n < \infty) = 1$ but

$\forall M : P(\sup X_n < M) < 1$

11) $\frac{X_n}{c_n} \to 0$ a.s

Know. $\forall n. \;\; \exists N(n)$

$P(|X_n| > N(n)) < 2^{-n}$

(Proof. If not. $\forall N : P(|X_n| > N) \geq 2^{-n}$.

$P(\bigcap_{N=1}^{\infty} \{|X_n| > N\}) \geq 2^{-n}$ — contradiction with $|X_n| < \infty$)

$c_n = 2^n N(n)$.

$\to P\left(\frac{|X_n|}{c_n} > 2^{-n}\right) = P(|X_n| > N(n)) < 2^{-n}$.

$P\left(\left\{\frac{|X_n|}{c_n} > 2^{-n} \;\; \text{i.o}\right\}\right) = 0$

$$P\left(\left\{\frac{|X_n|}{c_n} > 2^{-n} \quad i.o\right\}\right) = 0$$

A.S. $\exists N: n > N \quad \frac{|X_n|}{c_n} < 2^{-n}$

0

22) Hint:

$$\sum_{n=1}^{\infty} \min\left(P(A_n), P(A_n^c)\right) = \infty.$$

B_n: $B_n = A_n$ or A_n^c: $P(B_n) \geq \min\left(P(A_n), P(A_n^c)\right)$

$\sum P(B_n) = \infty$. B_n - independent.

$\sum P(B_n^c) = \infty$. $\left(P(B_n^c) \geq P(B_n)\right)$

26) Hint:

$\cancel{\sum P(E_n) < \infty}$

$\sum P(E_n \cap E_{n+1}^c) < \infty$

$\bigcap_{n=1}^{\infty}\left(\bigcup_{j=n}^{\infty} E_j\right)$ Want to estimate:

$\cancel{P\left(\bigcup_{j=n}^{\infty} E_j\right) \leq \sum_{j=n}^{\infty} P(E_j)} \to 0$

$$\bigcup_{j=n}^{\infty} E_j \overset{?}{\subseteq} \left(\left(E_n \cap E_{n+1}^c\right) \cup \left(E_{n+1} \cap E_{n+2}^c\right) \cup \dots \left(E_j \cap E_{j+1}^c\right)\right) \cup \bigcap_{j=n}^{\infty} E_j$$